First Order Partial Differential Equations

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Overview

We are concerned in the course with partial differential equations with one dependent variable z and mostly two independent variables x and y.

We discuss the following in three lectures.

- Formation of partial differential equations by either the elimination of arbitrary constants or by the elimination of arbitrary functions.
- Solution of partial differential equation by direct integration.
- Standard types of first order partial differential equations.

Introduction

Many physical and engineering problem can be interpreted by the ideas in differential equations. Newton's governing equations describing mechanical systems, Maxwell's equations describing electro dynamical phenomena and Schrodinger's equations describing the aspects of quantum mechanics are some examples to be mentioned.

Especially, propagation of waves, heat conduction and many other physical phenomena can be explained by partial differential equations.

A **partial differential equation** is the one in which there is one **dependent variable** which is the function of two or more independent variables with its partial derivatives. Most of the mathematical models in partial differential equation have two independent variables. Convention is that we take the variable z as a dependent variable and x and y as independent variables so the relation is $z = f(x, y)$.

In general, initial conditions or boundary conditions are given to describe the model completely. The order of a partial differential equation is the order of the highest partial differential coefficient occurring in it.

By a solution of a partial differential equation, we mean the expression of the form $z = f(x, y)$ which upon proper partial differentiation, coincides with the given partial differential equation on the same domain. Implicit in this idea of solution is the stipulation that z possesses as many continuous partial derivatives as required by the partial differential equation.

For example, a solution to a second order equation should have two continuous partial derivatives so that it makes sense to calculate the derivatives and substitute them in the given equation. Whereas the solution of an ordinary differential equation involves arbitrary constants, the general solution of a partial differential equation involves arbitrary functions.

Formation of partial differential equations

The partial differential equation of a given relation can be formed by either the elimination of the arbitrary constants or by the elimination of arbitrary functions depending upon whether the given relation involved by the arbitrary constants or arbitrary functions.

We concentrate the partial differential equation formed by relation having one dependent variable namely z and two independent variables namely x and y. This relation may be of arbitrary functions or arbitrary constants.

We use the following notions for the convenience which are used globally.

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

Formation of partial differential equations by eliminating arbitrary constants

Let $f(x, y, z, a, b) = 0$ be a given relation with two arbitrary constants a and b. Differentiating the above equation partially with respect to x and y we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \implies \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0$$
$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \implies \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0.$$

Eliminating a and b between the above three equations, we get a partial differential equation of the form

$$F(x, y, z, p, q) = 0.$$

Examples

Example 1.

1. *Form the partial differential equation by eliminating the constants a and b from* $z = (x + a)(y + b)$. *[Answer :* $z = pq$*]*
2. *Form the partial differential equation by eliminating the constants a and b from* $z = ax^n + by^n$. *[Answer:* $nz = px + qy$*]*
3. *Form the partial differential equation by eliminating the constants a and b from* $z = (x^2 + a^2)(y^2 + b^2)$. *[Answer:* $4xyz = pq$*]*
4. *Form the partial differential equation by eliminating the constants a and b from* $z = ax + by + \sqrt{a^2 + b^2}$.
[Answer: $z = px + qy + \sqrt{p^2 + q^2}$*]*
5. *Form the partial differential equation by eliminating the constants from* $\log(az - 1) = x + ay + b$. *[Answer:* $q(z - p) = p$*]*

Example 2.

1. *Form the partial differential equation of the surface* $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$, *where* a *and* b *are constants.* *[Answer:* $p^2+q^2=\tan^2\alpha$*]*
2. *Find the partial differential equation of all spheres whose center lies on z-axis. [Hint : If the equation of the sphere whose center lies on z-axis is given by* $x^2+y^2+(z-c)^2=r^2$, *where* c *and* r *are constants, then PDE is* $xq=yp$.*]*
3. *Find the partial differential equation of all spheres whose radius is unity and center lies on* xy *plane. [Hint : If the equation of the sphere whose radius is unity and center lies on* xy *plane is given by* $(x-a)^2+(y-b)^2+z^2=1$, *where* a *and* b *are constants, then* $z^2(p^2+q^2+1)=1$.*]*

Example 3.

1. *Find the partial differential equation of all planes passing through the origin.*
   [Hint : The equation of the plane which passes through origin is given by $ax + by + cz = 0$*, where* a, b *and* c *are constants, then* $z = px + qy$*]*
2. *Find the partial differential equation of all planes whose sum of* x, y, z *intercepts is unity.*
   [Hint : The equation of the plane with a, b, c *as the* x, y, z *intercepts is* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$*. Answer:* $z = p\ x + q\ y + \frac{pq}{p+q-pq}$*.]*

Formation of partial differential equations by eliminating arbitrary functions

Formation of partial differential equations by the elimination of arbitrary functions from the given relation is explained in the following examples.

The elimination of one arbitrary functions from the given relation gives partial differential equations of first order and elimination of two arbitrary functions from the given relation gives partial differential equations of second or higher order.

Example 4.

1. *Form the partial differential equation by eliminating the arbitrary function f from* $z = f(x+y)$. *[Answer:* $p = q$*]*
2. *Form the partial differential equation by eliminating f from* $z = f(x^2 + y^2)$. *[Answer:* $py = qx$*]*
3. *Form the partial differential equation by eliminating f from* $z = x + y + f(xy)$. *[Answer:* $px - qy = x - y$*]*
4. *Form the partial differential equation by eliminating f from* $z = f(x^2 + y^2 + z^2)$. *[Answer:* $yp = xq$*]*
5. *Form the partial differential equation by eliminating f from* $z = x^2 + 2f\left(\frac{1}{y} + \log\ x\right)$. *[Answer:* $px + qy^2 = 2x^2$*]*
6. *Form the partial differential equations by eliminating the arbitrary functions from* $z = f_1(x) + f_2(y)$. *[Answer:* $\frac{\partial^2 z}{\partial y \partial x} = 0$*]*

Example 5.

1. *Form the partial differential equations by eliminating the arbitrary functions from* $z = yf_1(x) + f_2(y)$. *[Answer:* $\frac{\partial z}{\partial x} = y\frac{\partial^2 z}{\partial y \partial x}$*]*
2. *Form the partial differential equation by eliminating the* f *and* ϕ *from* $z = f(y) + \phi(x + y + z)$. *[Answer:* $r(1 + q) = s(1 + p)$*]*
3. *Form a partial differential equation from the relation by eliminating the arbitrary functions* f_1 *and* f_2 *from* $z = f_1(x + 2y) + f_2(x - 2y)$. *[Answer:* $4\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$.*]*
4. *Form a PDE from the relation* $z = f_1(ax + by) + f_2(cx + dy)$. *[Answer:* $bd\frac{\partial^2 z}{\partial x^2} - (ad + bc)\frac{\partial^2 z}{\partial x \partial y} + ac\frac{\partial^2 z}{\partial y^2} = 0$*]*
5. *Form a partial differential equation from the relation* $z = f_1(2x + 3y) + f_2(4x + 5y)$. *[Answer:* $15\frac{\partial^2 z}{\partial x^2} - 22\frac{\partial^2 z}{\partial x \partial y} + 8\frac{\partial^2 z}{\partial y^2} = 0$*]*
6. *Form a partial differential equation by eliminating the functions from* $z = f_1(x^2 + 3y) + f_2(x^2 - 3y)$ *[Answer :* $9\frac{\partial^2 z}{\partial x^2} - \frac{9}{x}\frac{\partial z}{\partial x} - 4x^2\frac{\partial^2 z}{\partial y^2} = 0$*]*

Formation of partial differential equations by elimination of arbitrary function ϕ from $\phi(u, v) = 0$ where u and v are functions of x, y and z.

Let

$$\phi(u, v) = 0 \tag{1}$$

be the given relation with u and v are functions of x, y and z.

Differentiating (1) partially with respect to x and y we get

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0. \tag{3}$$

Now eliminating ϕ between (2) and (3) we get the required partial differential equation as

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

Example 6.

1. *Form a partial differential equation by eliminating* ϕ *from* $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$. *[Answer: $z = xp + yq$ is the required solution.]*
2. *Form a partial differential equation by eliminating* ϕ *from* $\phi(x+y+z, x^2+y^2+z^2) = 0$. *[Answer:$(y-z)p + (z-x)q = x - y$]*

Solution of partial differential equations

Consider the relation

$$z = ax + by. \tag{1}$$

Eliminating the arbitrary constants a and b we get

$$z = px + qy. \tag{2}$$

Also consider the relation

$$z = x\ f\left(\frac{y}{x}\right). \tag{3}$$

Eliminating the arbitrary function f we get

$$z = px + qy.$$

From these we conclude that both (1) and (3) are solutions of the same equation (2) and for a single partial differential equation we have more than one solutions.

Types of solutions

1. A solution which contains the number of arbitrary constants is equal to the number of independent variables is called **complete solution** or **complete integral**.
2. A complete solution in which if a particular value is given to arbitrary constant is called **particular solution**.
3. A solution which contains the maximum possible number of arbitrary functions is called a **general solution.**

Simple partial differential equations can be solved by direct integration. The method of solving such equations explained as follows.

Example 7.

Solve $\frac{\partial z}{\partial x} = 0$.

Solution. *The given equation is*

$$\frac{\partial z}{\partial x} = 0.$$

This shows that z *is a function of the independent* x. *Integrating with respect to* x

$$z = f(y)$$

is the solution of the given partial differential equation.

Example 8.

Solve $\frac{\partial^2 z}{\partial x \partial y} = 0$.

Solution. *The given equation can be written as*

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0.$$

Integrating with respect to x

$$\frac{\partial z}{\partial y} = f_1(y).$$

Now integrating with respect to y

$$\begin{aligned} z &= \int f_1(y)dy + f_2(x) \\ &= g_1(y) + f_2(x). \end{aligned}$$

Hence the solution is $z = g_1(y) + f_2(x)$.

Example 9.

Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + c$.

Solution. *The given equation can be written as*

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{x}{y} + c.$$

Integrating with respect to x

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + cx + f_1(y).$$

Now integrating with respect to y

$$\begin{aligned} z &= \frac{x^2}{2}\log y + cxy + \int f_1(y)dy + f_2(x) \\ &= \frac{x^2}{2}\log y + cxy + g_1(y) + f_2(x). \end{aligned}$$

Hence the solution is $z = \frac{x^2}{2}\log y + cxy + g_1(y) + f_2(x)$.

Example 10.

Solve $\frac{\partial^2 z}{\partial x^2} = \cos x$.

Solution. *The given equation is*

$$\frac{\partial^2 z}{\partial x^2} = \cos x.$$

Integrating with respect to x

$$\frac{\partial z}{\partial x} = \sin x + f_1(y).$$

Again integrating with respect to x

$$z = -\cos x + x f_1(y) + f_2(y).$$

Example 11.

Solve $\frac{\partial^2 z}{\partial y \partial x} = e^{-x} \cos\ y$ *given that* $z = 0$ *when* $x = 0$ *and* $\frac{\partial z}{\partial x} = 0$ *when* $y = 0$.
Solution. *The given equation can be written as*

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = e^{-x} \cos\ y.$$

Integrating with respect to y, $\frac{\partial z}{\partial x} = e^{-x} \sin\ y + f_1(x)$.
Applying $\frac{\partial z}{\partial x} = 0$ *when* $y = 0$, *we get* $f_1(x) = 0$, *hence* $\frac{\partial z}{\partial x} = e^{-x} \sin\ y$.
Now integrating with respect to x

$$z = -e^{-x} \sin y + f_2(y).$$

Applying $z = 0$ *when* $x = 0$

$$0 = -\sin\ y + f_2(y) \Rightarrow f_2(y) = \sin y.$$

Hence

$$z = -e^{-x} \sin\ y + \sin\ y.$$

Standard types of first order partial differential equations

Type 1. Consider the partial differential equation of the form

$$F(p, q) = 0. \tag{1}$$

Here the variables x, y, z do not occur explicitly. Let

$$z = ax + by + c. \tag{2}$$

be a solution of the given equation. Then

$$\frac{\partial z}{\partial x} = p = a, \quad \frac{\partial z}{\partial y} = q = b.$$

Substituting these values of p and q in the equation (1) we get $F(a, b) = 0$. Solving for b we get $b = f(a)$ then

$$z = ax + f(a)y + c \tag{3}$$

is the complete solution of the given equation.

Type 1

To get the general solution put $c = g(a)$

$$z = ax + f(a)y + g(a) \tag{4}$$

and differentiating with respect to a

$$0 = x + f'(a)y + g'(a). \tag{5}$$

Eliminating a between (4) and (5) we get the general solution.

To get the singular solution, differentiating (2) with respect to c we get $0 = 1$, which is not possible. Hence there is no singular solution.

Example 12.

Solve $p^2 + p - q^2 = 0$.

Solution. *The given equation is of the form* $F(p, q) = 0$. *The complete solution is*

$$z = ax + by + c \tag{1}$$

where $a^2 + a - b^2 = 0$. *Solving for* b *we get* $b^2 = a^2 + a, \quad \text{so} \quad b = \sqrt{a^2 + a}$. *Hence the complete solution is*

$$z = ax + \sqrt{a^2 + a}y + c. \tag{2}$$

Differentiating (2) *with respect to* c *we get* $0 = 1$, *which is not possible. Hence there is no singular solution. To get the general solution put* $c = g(a)$

$$z = ax + \sqrt{a^2 + a}y + g(a) \tag{3}$$

and differentiating with respect to a, *we have* $0 = x + \frac{1+2a}{2\sqrt{a^2+a}}y + g'(a)$. *Eliminating* a *between* (3) *and* (4) *we get the general solution.*

Example 13.

Solve $p^2 + q^2 = 9$.

Solution. *The given equation is of the form* $F(p, q) = 0$. *The complete solution is*

$$z = ax + by + c \tag{1}$$

where $a^2 + b^2 = 9$. *Solving for* b *we get* $b^2 = 9 - a^2$, *so* $b = \sqrt{9 - a^2}$. *Hence the complete solution is*

$$z = ax + \sqrt{9 - a^2}y + c \tag{2}$$

Differentiating (2) *with respect to* c *we get* $0 = 1$, *which is not possible. Hence there is no singular solution. To get the general solution put* $c = g(a)$

$$z = ax + \sqrt{9 - a^2}y + g(a) \tag{3}$$

and differentiating with respect to a

$$0 = x + \frac{-2a}{2\sqrt{9 - a^2}}y + g'(a). \tag{4}$$

Eliminating a *between* (3) *and* (4) *we get the general solution.*

Example 14.

Solve $pq + p + q = 0$.

Solution. *The given equation is of the form* $F(p, q) = 0$. *The complete solution is*

$$z = ax + by + c \tag{1}$$

where $ab + a + b = 0$. *Solving for* b *we get* $b(1 + a) = -a$, *so* $b = -\frac{a}{1+a}$. *Hence the complete solution is*

$$z = ax - \frac{a}{1+a}y + c. \tag{2}$$

Differentiating (2) *with respect to* c *we get* $0 = 1$, *which is not possible. Hence there is no singular solution. To get the general solution put* $c = g(a)$

$$z = ax - \frac{a}{1+a}y + g(a) \tag{3}$$

and differentiating with respect to a

$$0 = x - \frac{1}{(1+a)^2}y + g'(a). \tag{4}$$

Eliminating a *between* (3) *and* (4) *we get the general solution.*

Type 2

Consider the partial differential equations of the form

$$z = px + qy + F(p, q).$$

This equation is called the **Clairut's equation**.

Let

$$z = ax + by + c$$

be a solution of the given equation. Then

$$\frac{\partial z}{\partial x} = p = a,$$
$$\frac{\partial z}{\partial y} = q = b.$$

Substituting these values of p and q in the given equation, we get

$$z = ax + by + F(a, b).$$

Example 15.

Solve $z = px + qy + pq$.

Solution. *The given equation is of the form* $z = px + qy + F(p, q)$. *The complete solution is*

$$z = ax + by + ab. \tag{1}$$

To find the singular solution differentiating (1) *partially with respect to* a

$$0 = x + b \Rightarrow b = -x$$

and differentiating (1) *partially with respect to* b *we get*

$$0 = y + a \Rightarrow a = -y.$$

Substituting these values in (1) *we get*

$$z = -(y)x - (x)y + (-y)(-x)$$
$$z = -xy - xy + xy$$
$$x = -xy$$
$$z + xy = 0.$$

Example 16.

Solve $z = px + qy + p^2 + q^2$

Solution. *The given equation is of the from* $z = px + qy + F(p, q)$. *The complete solution is*

$$z = ax + by + a^2 + b^2 \qquad (1)$$

To find the singular solution differentiating (1) *partially with respect to* a

$$0 = x + 2a \Rightarrow a = -\frac{x}{2}$$

and differentiating (1) *partially with respect to* b *we get,* $0 = y + 2b \Rightarrow b = -\frac{y}{2}$. *Substituting these values in* (1) *we get*

$$\begin{aligned} z &= -\left(\frac{x}{2}\right)x - \left(\frac{y}{2}\right)y + \left(-\frac{x}{2}\right)^2 + \left(-\frac{y}{2}\right)^2 \\ z &= -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4} \\ z &= -\frac{x^2}{4} - \frac{y^2}{4} \\ 4z &= -(x^2 + y^2). \end{aligned}$$

Example 17.

Solve $z = px + qy + p^2 + pq + q^2$.

Solution. *The given equation is of the form* $z = px + qy + F(p, q)$.
The complete solution is

$$z = ax + by + a^2 + ab + b^2. \qquad (1)$$

To find the singular solution differentiating (1) *partially with respect to a and b we get,*
$x + 2a + b = 0$ *and* $y + a + 2b = 0$.
Solving we get $a = \frac{1}{3}(y - 2x)$ *and* $b = \frac{1}{3}(x - 2y)$. *Substituting these values of a and b in* (1) *we get*

$$z = \frac{1}{3}(y-2x)x + \frac{1}{3}(x-2y)y + \frac{1}{9}(y-2x)^2 + \frac{1}{9}(y-2x)(x-2y) + \frac{1}{9}(x-2y)^2$$

$$9z = 3x(y-2x) + 3y(x-2y) + (y-2x)^2 + (y-2x)(x-2y) + (x-2y)^2$$

$$9z = -3x^2 + 3xy - 3y^2.$$

Hence $3z + x^2 - xy + y^2 = 0$.

Example 18.

Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Solution. *The given equation is of the form* $z = px + qy + F(p, q)$.
The complete solution is

$$z = ax + by + \sqrt{1 + a^2 + b^2}. \tag{1}$$

To find the singular solution differentiating (1) *partially with respect to a and b we get* $x = -\frac{a}{\sqrt{1+a^2+b^2}}$ *and* $y = -\frac{b}{\sqrt{1+a^2+b^2}}$. *Hence* $\sqrt{1 - x^2 - y^2} = \frac{1}{\sqrt{1+a^2+b^2}}$. *From x and y, we get* $a = -\frac{x}{\sqrt{1-x^2-y^2}}$ *and* $b = -\frac{y}{1-x^2-y^2}$. *Substituting the values in* (1) *we have*

$$\begin{aligned} z &= -\frac{x^2}{\sqrt{1 - x^2 - y^2}} - \frac{y^2}{\sqrt{1 - x^2 - y^2}} + \frac{1}{\sqrt{1 - x^2 - y^2}} \\ z &= \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}} \\ z^2 &= 1 - x^2 - y^2. \end{aligned}$$

Thus $x^2 + y^2 + z^2 = 1$.

Type 3.

This type of partial differential equation can be further classified into three categories. We discuss each case as follows.

Case I. Consider the partial differential equation of the form

$$F(x, p, q) = 0.$$

Let us assume that $q = a$. Then

$$F(x, p, a) = 0.$$

Solving for p we get

$$p = \phi(x, a).$$

Since z is a function of x and y

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= pdx + qdy \\ &= \phi(x, a) dx + qdy \\ z &= \int \phi(x, a) dx + \int ady + c \\ z &= \int \phi(x, a) dx + ay + c. \end{aligned}$$

Example 19.

Solve $pq = x$.

Solution. *The given equation is of the form* $F(x, p, q) = 0$.
Let us assume that $q = a$. *Then* $pa = x \Rightarrow p = \frac{x}{a}$.

$$\begin{aligned} dz &= pdx + qdy \\ &= \frac{x}{a}dx + a\ dy \\ z &= \int \frac{x}{a}dx + \int\ ady + c \\ z &= \frac{x^2}{2a} + ay + c. \end{aligned}$$

Example 20.

Solve $q = px + p^2$.

Solution. *The given equation is of the form* $F(x, p, q) = 0$.
Let us assume that $q = a$. *then*

$$
\begin{aligned}
p^2 + px - a &= 0 \\
p &= \frac{-x \pm \sqrt{x^2 + 4a}}{2} \\
dz &= pdx + qdy \\
&= \left(\frac{-x \pm \sqrt{x^2 + 4a}}{2} \right) dx + a\, dy \\
z &= \int \frac{x}{2} dx \pm \frac{1}{2} \int \sqrt{x^2 + 4a}\, dx + \int ady + c \\
&= \frac{x^2}{4} \pm \frac{1}{2} \left[\frac{x}{2} \sqrt{x^2 + 4a} + \frac{4a}{2} \sinh^{-1} \left(\frac{x}{2\sqrt{a}} \right) \right] + ay + c \\
&= \frac{x^2}{4} \pm \left[\frac{x}{4} \sqrt{x^2 + 4a} + a \sinh^{-1} \left(\frac{x}{2\sqrt{a}} \right) \right] + ay + c.
\end{aligned}
$$

Case II

Consider the partial differential equations of the form

$$F(y, p, q) = 0.$$

Let us assume that $p = a$. Then

$$F(y, a, q) = 0.$$

Solving for q we get

$$q = \phi(y, a)$$

Since z is a function of x and y

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= pdx + qdy \\ &= adx + \phi(y, a)dy \\ z &= \int adx + \int \phi(y, a)dy + c \\ z &= ax + \int \phi(y, a)dy + c. \end{aligned}$$

Example 21.

Solve $q^2 = yp^3$.

Solution. *The given equation is of the form* $F(y, p, q) = 0$.
Let us assume that $p = a$. *Then*

$$q^2 = ya^3 = 0$$

$$q = a^{\frac{3}{2}}\sqrt{y}.$$

$$\begin{aligned} dz &= pdx + qdy \\ &= a\ dx + a^{\frac{3}{2}}\sqrt{y}dy \\ z &= \int a\ dx + \int a^{\frac{3}{2}}\sqrt{y}\ dy + c \\ x &= a\ x + \frac{2}{3}a^{\frac{3}{2}}y^{\frac{3}{2}} + c. \end{aligned}$$

Example

Example 22.

Solve $p = qy + q^2$.

Solution. *The given equation is of the form* $F(y, p, q) = 0$.
Let us assume that $p = a$. *Then*

$$
\begin{aligned}
q^2 + qy - a &= 0 \\
q &= \frac{-y \pm \sqrt{y^2 + 4a}}{2} \\
dz &= pdx + qdy \\
&= a\ dx + \left(\frac{-y \pm \sqrt{y^2+4a}}{2} \right) dy \\
z &= \int adx + \int \frac{y}{2} dy \pm \frac{1}{2} \int \sqrt{y^2+4a}\ dy + c \\
&= ax + \frac{y^2}{4} \pm \frac{1}{2} \left[\frac{y}{2} \sqrt{y^2+4a} + \frac{4a}{2} \sinh^{-1}\left(\frac{y}{2\sqrt{a}} \right) \right] + c \\
&= ax + \frac{y^2}{4} \pm \left[\frac{y}{4} \sqrt{y^2+4a} + a \sinh^{-1}\left(\frac{y}{2\sqrt{a}} \right) \right] + c
\end{aligned}
$$

Case III

Consider the partial differential equations of the form

$$F(z, p, q) = 0.$$

Let us assume that $q = ap$. Then

$$F(z, p, ap) = 0.$$

Solving for p we get

$$\begin{aligned} p &= \phi(z, a) \\ q &= ap = a\phi(z, a) \end{aligned}$$

Since z is a function of x and y

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= pdx + qdy \\ &= \phi(z, a)dx + a\phi(z, a)dy \\ \int \frac{dz}{\phi(z, a)} &= \int dx + \int ady + c \\ \int \frac{dz}{\phi(z, a)} &= x + ay + c. \end{aligned}$$

Example 23.

Solve $1 + z^3 = z^4 pq$.

Solution. *The given equation is of the form* $F(z,p,q) = 0$. *Let us assume that* $q = ap$. *Then* $1 + z^3 = z^4 p^2 a$ *gives* $p = \frac{1}{\sqrt{a}} \frac{\sqrt{1+z^3}}{z^2}$ *and* $q = a\frac{1}{\sqrt{a}} \frac{\sqrt{1+z^3}}{z^2}$. *So* $dz = pdx + qdy$ *gives*

$$\int dz = \frac{1}{\sqrt{a}} \int \frac{\sqrt{1+z^3}}{z^2} dx + a\frac{1}{a} \int \frac{\sqrt{1+z^3}}{z^2} dy$$

$$\int \frac{z^2}{\sqrt{1+z^3}} dz = \frac{1}{\sqrt{a}} \left(\int dx + a \int dy + c \right)$$

$$\frac{2}{3}\sqrt{1+z^3} = \frac{1}{\sqrt{a}}(x + ay + c)$$

$$\frac{4}{9}(1+z^3) = \frac{1}{a}(x + ay + c)^2$$

$$4\ a(1+z^3) = 9(x + ay + c)^2.$$

Example 24.

Solve $z = p^2 + q^2$.

Solution. *The given equation is of the form* $F(z, p, q) = 0$. *Let us assume that* $q = ap$. *Then* $z = p^2 + a^2p^2$ *gives* $p = \frac{1}{\sqrt{1+a^2}}\sqrt{z}$ *and* $q = \frac{a}{\sqrt{1+a^2}}\sqrt{z}$. *So* $dz = pdx + qdy$ *gives*

$$\int dz = \frac{1}{\sqrt{1+a^2}} \int \sqrt{z}dx + \frac{a}{\sqrt{1+a^2}} \int \sqrt{z}dy$$

$$\int \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}} \left(\int dx + a \int dy + c \right)$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}(x + ay + c)$$

$$4z = \frac{1}{1+a^2}(x + ay + c)^2.$$

Example 25.

Solve $p(1-q^2) = q(1-z)$.

Solution. *The given equation is of the form* $F(z,p,q) = 0$. *Let us assume that* $q = ap$. *Then*

$$\begin{aligned}
p(1-a^2p^2) &= ap(1-z)\\
(1-a^2p^2) &= a(1-z)\\
a^2p^2 &= 1-a+az\\
p &= \frac{1}{a}\sqrt{1-a+az}\\
q &= \sqrt{1-a+az}\\
dz &= pdx+qdy\\
\int dz &= \frac{1}{a}\int \sqrt{1-a+az}\,dx + \int \sqrt{1-a+az}\,dy\\
\int \frac{dz}{\sqrt{1-a+az}} &= \frac{1}{a}\left(\int dx + a\int dy + c\right)\\
\frac{2}{a}\sqrt{1-a+az} &= \frac{1}{a}(x+ay+c)\\
4(1-a+az) &= (x+ay+c)^2.
\end{aligned}$$

Example 26.

Solve $q^2 = z^2p^2(1-p^2)$.

Solution. *The given equation is of the form* $F(z,p,q)=0$. *Let us assume that* $q = ap$. *Then*

$$\begin{aligned} a^2p^2 &= z^2p^2(1-p^2) \\ a^2 &= z^2(1-p^2) \\ z^2p^2 &= z^2 - a^2 \\ p &= \frac{1}{z}\sqrt{z^2-a^2} \\ q &= a\frac{1}{z}\sqrt{z^2-a^2}. \end{aligned}$$

$$\begin{aligned} dz &= pdx + qdy \\ \int dz &= \int \frac{1}{z}\sqrt{z^2-a^2}dx + \int a\frac{1}{z}\sqrt{z^2-a^2}dy \\ \int \frac{z\ dz}{\sqrt{z^2-a^2}} &= \left(\int dx + a\int dy + c\right) \\ \sqrt{z^2-a^2} &= (x+ay+c) \\ z^2-a^2 &= (x+ay+c)^2 \\ z^2 &= a^2+(x+ay+c)^2. \end{aligned}$$

Example

Example 27.

Solve $p^3 + q^3 = z^3$.

Solution. *The given equation is of the form* $F(z, p, q) = 0$. *Let us assume that* $q = ap$. *Then*

$$p = \frac{1}{(1+a^3)^{\frac{1}{3}}} z$$

$$q = a\frac{1}{(1+a^3)^{\frac{1}{3}}} z$$

$$dz = pdx + qdy$$

$$\int dz = \int \frac{1}{(1+a^3)^{\frac{1}{3}}} z\ dx + \int a\frac{1}{(1+a^3)^{\frac{1}{3}}} z\ dy$$

$$(1+a^3)^{\frac{1}{3}} \int \frac{dz}{z} = \left(\int dx + a + \int dy + c\right)$$

$$(1+a^3)^{\frac{1}{3}} \log\ z = (x + ay + c).$$

Example 28.

Solve $z^2(p^2+q^2+1)=1$.

Solution. *The given equation is of the form* $F(z,p,q)=0$. *Let us assume that* $q=ap$. *Then*

$$p=\frac{1}{\sqrt{1+a^2}}\frac{\sqrt{1-z^2}}{z}$$
$$q=\frac{a}{\sqrt{1+a^2}}\frac{\sqrt{1-z^2}}{z}.$$

$$dz=pdx+qdy$$
$$\int dz=\int \frac{1}{\sqrt{1+a^2}}\frac{\sqrt{1-z^2}}{z}dx+\int a\,\frac{1}{\sqrt{1+a^2}}\frac{\sqrt{1-z^2}}{z}dy$$
$$\int\frac{z\,dz}{\sqrt{1-z^2}}=\frac{1}{\sqrt{1+a^2}}\left(\int dx+a\int dy+c\right)$$
$$-\sqrt{1-z^2}=\frac{1}{\sqrt{1+a^2}}(x+ay+c).$$

Example 29.

Solve $2p^2 - qz = z^2$.

Solution. *The given equation is of the form* $F(z, p, q) = 0$. *Let us assume that* $q = ap$. *Then*

$$p = z\left(\frac{a \pm \sqrt{a^2+8}}{2}\right)$$

$$q = a\ z\left(\frac{a \pm \sqrt{a^2+8}}{2}\right).$$

$$dz = pdx + qdy$$

$$\int dz = \int z\left(\frac{a \pm \sqrt{a^2+8}}{2}\right)dx + \int a\ z\ \left(\frac{a \pm \sqrt{a^2+8}}{2}\right)dy$$

$$\int \frac{dz}{z} = \left(\frac{a \pm \sqrt{a^2+8}}{2}\right)(\int dx + a\int dy + c)$$

$$\log\ z = \left(\frac{a \pm \sqrt{a^2+8}}{2}\right)(x + ay + c).$$

Example 30.

Solve $p(1+q) = q\ z$.

Solution. *The given equation is of the form* $F(z,p,q) = 0$. *Let us assume that* $q = ap$. *Then*

$$\begin{aligned} p(1+ap) &= zpa \\ p[(1+az)+ap] &= 0 \\ p &= 0, \frac{1}{a}(az-1) \\ q &= a\frac{1}{a}(az-1). \end{aligned}$$

$$\begin{aligned} dz &= pdx + qdy \\ \int dz &= \frac{1}{a}\int (az-1)dx + a\frac{1}{a}\int (az-1)dy \\ \int \frac{1}{az-1}dz &= \frac{1}{a}\left(\int dx + a\int dy + c\right) \\ \frac{1}{a}\log(az-1) &= \frac{1}{a}(x+ay+c) \\ \log(az-1) &= (x+ay+c). \end{aligned}$$

Type 4

Consider the partial differential equation of the form

$$F_1(x, p) = F_2(y, q).$$

Let us assume that

$$\begin{aligned} F_1(x, p) &= a \\ F_2(y, q) &= a. \end{aligned}$$

Solving for p and q we get

$$\begin{aligned} p &= \phi_1(x, a) \\ q &= \phi_2(y, a). \end{aligned}$$

Since z is a function of x and y

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= pdx + qdy \\ &= \phi_1(x, a)dx + \phi_2(y, a)dy \\ z &= \int \phi_1(x, a)dx + \int \phi_2(y, a)dy + c. \end{aligned}$$

Example 31.

Solve $p + q = x + y$.

Solution. *The given equation is of the form* $F_1(x, p) = F_2(y, q)$.

$$
\begin{aligned}
p + q = x + y &= a(say) \\
p - x = a &\Rightarrow p = x + a \\
q - y = a &\Rightarrow q = y + a \\
dz &= pdx + qdy \\
z &= \int (x + a)dx + \int (y + a)dy + c \\
z &= \frac{(x + a)^2}{2} + \frac{(y + a)^2}{2} + c.
\end{aligned}
$$

Example 32.

Solve $p^2 + q^2 = x + y$.

Solution. *The given equation is of the form* $F_1(x, p) = F_2(y, p)$.

$$p^2 + q^2 = x + y = a(say)$$
$$p^2 - x = a \Rightarrow p = \sqrt{x + a}$$
$$y - q^2 = a \Rightarrow q = \sqrt{y - a}$$
$$dz = pdx + qdy$$
$$z = \sqrt{x + a}dx + \int \sqrt{y - a}dy + c$$
$$z = \frac{2}{3}(x + a)^{\frac{3}{2}} + \frac{2}{3}(y - a)^{\frac{3}{2}} + c.$$

Example 33.

Solve $p^2 + q^2 = x^2 + y^2$.
Solution. *The given equation is of the form* $F_1(x, p) = F_2(y, q)$.
From $p^2 + q^2 = x^2 + y^2 = a^2\,(say)$,
we have,

$$
\begin{aligned}
p^2 - x^2 = a^2 &\Rightarrow p = \sqrt{x^2 + a^2}, \\
y^2 - q^2 = a^2 &\Rightarrow q = \sqrt{y^2 - a^2} \\
dz &= pdx + qdy \\
z &= \int \sqrt{x^2 + a^2}dx + \int \sqrt{y^2 - a^2}dy + c \\
z &= \frac{x}{2}\sqrt{x^2 + a^2}\frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c.
\end{aligned}
$$

Example

Example 34.

Solve $q^2x(1+y^2) = py^2$.

Solution. *The given equation can written in form* $F_1(x,p) = F_2(y,q)$.

$$\begin{aligned}
q^2x(1+y^2) &= py^2 \\
q^2\frac{1+y^2}{y^2} &= \frac{p}{x} = a^2 (say) \\
\frac{p}{x} &= a^2 \Rightarrow p = a^2x \\
q^2\frac{1+y^2}{y^2} &= a^2 \Rightarrow q = \frac{ay}{\sqrt{1+y^2}} \\
dz &= pdx + qdy \\
z &= \int a^2x \; dx + \int \frac{ay}{\sqrt{1+y^2}} dy + c \\
z &= a^2\frac{x^2}{2} + a\sqrt{1+y^2} + c.
\end{aligned}$$

Type 5

Consider the partial differential equations of the form

$$F(x^m p, y^n q) = 0 \tag{1}$$

and

$$F(z, x^m p, y^n q) = 0, \tag{2}$$

where m and n are constants.

Case I. For $m \neq 1$ and $n \neq 1$ put

$$X = x^{1-m} \Rightarrow \frac{\partial X}{\partial x} = (1-m)x^{-m}, \quad Y = y^{1-n} \Rightarrow \frac{\partial Y}{\partial y} = (1-n)y^{-n}$$

$$P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P(1-m)x^{-m}$$

$$\Rightarrow P(1-m) = px^m \tag{3}$$

$$Q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = Q(1-n)y^{-n}$$

$$\Rightarrow Q(1-n) = qy^n \tag{4}$$

where $P = \frac{\partial x}{\partial X}$ and $Q = \frac{\partial z}{\partial Y.}$ Using expressions (3) and (4), the equations (1) and (2) can be transformed into equations $F(P, Q) = 0$ and $F(z, P, Q) = 0$, respectively. These equations are easily solvable by standard method.

Case 2.

For $m = 1$ and $n = 1$ the equation is of the form $F(xp, yp) = 0$. To solve this put

$$\begin{aligned}
X &= \log x \Rightarrow \frac{\partial X}{\partial x} = \frac{1}{x} \\
Y &= \log y \Rightarrow \frac{\partial Y}{\partial y} = \frac{1}{y} \\
p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\frac{1}{x} \\
&\Rightarrow P = px \qquad (5) \\
q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = Q\frac{1}{y} \\
&\Rightarrow Q = qy \qquad (6)
\end{aligned}$$

where $P = \frac{\partial z}{\partial X}$ and $Q = \frac{\partial z}{\partial Y}$.

Substituting the values of (5) and (6) in (1) and (2) we get transformed equatons $F(P, Q) = 0$ and $F(z, P, Q) = 0$, respectively. These equations are easily solvable by standard method.

Example 35.

Solve $xp + yq = 1$.

Solution. *The given equation is of the form* $F(x^m p, y^n q) = 0$. *Here* $m = 1$ *and* $n = 1$. *Put* $X = \log x$ *and* $Y = \log y$, *then* $xp = P \quad yp = Q$. *Hence the equation becomes*

$$P + Q = 1$$

which is of the form $F(P, Q) = 0$.

$$Let\ z = aX + bY + c$$

be the solution with

$$\begin{aligned} a + b = 1 &\Rightarrow b = 1 - a \\ z &= aX + (1 - a)Y + c \\ z = a \ \log \ x &+ (1 - a)\log y + c. \end{aligned}$$

Example 36.

Find the solution of $px^2 + qy^2 = z^2$.

Solution. *The given equation is of the form* $F(x^m p, y^n q, z) = 0$. *Here* $m = 2$ *and* $n = 2$. *Put* $x^{1-2} = X$ *and* $y^{1-2} = Y$, *then* $x^2 p = -P, \quad y^2 q = -Q$. *Hence the equation becomes* $P + Q = -z^2$. *Putting* $Q = aP$, *we get* $P + aP = -z^2 \Rightarrow P = -\frac{z^2}{(1+a)}$ *and* $Q = -\frac{a\, z^2}{(1+a)}$.

$$\begin{aligned}
dz &= PdX + QdY \\
&= -\frac{z^2}{(1+a)}dX - \frac{a}{(1+a)}z^2 dY \\
-\int \frac{dz}{z^2} &= \frac{1}{(1+a)}\left(\int dX + a\int dY\right) \\
\frac{1}{z} &= \frac{1}{(1+a)}(X + aY) + c \\
\frac{1}{z} &= \frac{1}{(1+a)}\left(\frac{1}{x} + a\,\frac{1}{y}\right) + c.
\end{aligned}$$

Example 37.

Solve $x^2y^2pq = z^2$.

Solution. *The given equation is of the form* $F(x^m p, y^n q, z) = 0$. *Here* $m = 2$ *and* $n = 2$. *Put* $x^{1-2} = X$ *and* $y^{1-2} = Y$, *then* $x^2p = -P$ *and* $y^2q = -Q$. *Hence the equation becomes* $PQ = z^2$. *Putting* $Q = a^2P$, *we get* $P^2a^2 = z^2 \Rightarrow P = \frac{z}{a}$ *and* $Q = a^2\frac{z}{a} = az$.

$$
\begin{aligned}
dz &= PdX + QdY \\
&= \frac{1}{a}zdX + azdY \\
\int \frac{dz}{z} &= \frac{1}{a}\left(\int dX + a \int dY\right) \\
\log z &= \frac{1}{a}(X + aY) + c \\
\log z &= \frac{1}{a}\left(\frac{1}{x} + a\frac{1}{y}\right) + c.
\end{aligned}
$$

Example 38.

Solve $\frac{p}{x^2} + \frac{q}{y^2} = z$.

Solution. *The given equation is of the form* $F(x^m p, y^n q, z) = 0$. *Here* $m = -2$ *and* $n = -2$. *Put* $x^{1-(-2)} = x^3 = X$ *and* $y^{1-(-2)} = y^3 = Y$, *then* $x^{-2}p = 3P$ *and* $y^{-2}q = 3Q$. *Hence the equation becomes* $3P + 3Q = z$. *Putting* $Q = aP$, *we get* $3P + 3aP = z \Rightarrow P = \frac{z}{3(1+a)}$ *and* $Q = a\frac{z}{3(1+a)} = \frac{a}{3(1+a)}z$.

$$\begin{aligned} dz &= PdX + QdY \\ &= \frac{1}{3(1+a)}zdX + \frac{a}{3(1+a)}zdY \\ \int \frac{dz}{z} &= \frac{1}{3(1+a)}\left(\int dX + a\int dY\right) \\ \log z &= \frac{1}{3(1+a)}(X + aY) + c \\ \log\ z &= \frac{1}{3(1+a)}(x^3 + ay^3) + c. \end{aligned}$$

Example 39.

Solve $x^4p^2 - yzq = z^2$.

Solution. *The given equation can be written as* $(x^2p)^2 - (yz)q = z^2$*. Putting* $x^{1-2} = X$ *and* $\log\ y = Y$*, we get* $x^2p = P$ *and* $yp = Q$*. Hence the equation becomes* $P^2 - Qz = z^2$*. This is the form* $F(P, Q, z) = 0$*. Putting* $Q = aP$*, we get*

$$P^2 - azP = z^2 \Rightarrow P = \frac{z(a \pm \sqrt{a^2+4})}{2}, \quad \textit{and} \quad Q = a\frac{z(a \pm \sqrt{a^2+4})}{2}.$$

$$\begin{aligned}
dz &= PdX + Qdy \\
&= \frac{z(a \pm \sqrt{a^2+4})}{2} dX + a\frac{z(a \pm \sqrt{a^2+4})}{2} dY \\
\int \frac{dz}{z} &= \frac{(a \pm \sqrt{a^2+4})}{2}\left(\int dX + a \int dY\right) \\
\log\ z &= \frac{(a \pm \sqrt{a^2+4})}{2}(X + aY) + c \\
\log\ z &= \frac{(a \pm \sqrt{a^2+4})}{2}\left(\frac{1}{x} + a\ log\ y\right) + c.
\end{aligned}$$

Type 6.

Consider the partial differential equations of the form

$$F(z^m p, z^m q) = 0 \tag{1}$$

and

$$F_1(x, z^m p) = F_2(y, z^m q) \tag{2}$$

where m is a constant.

Case I For $m \neq -1$ put $Z = z^{m+1}$, we get

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} \Rightarrow P = (m+1)z^m p \Rightarrow \frac{P}{m+1} = p \tag{3}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y} \Rightarrow Q = (m+1)z^m q \Rightarrow \frac{Q}{m+1} = q, \tag{4}$$

where $P = \frac{\partial Z}{\partial x}$ and $Z = \frac{\partial Z}{\partial y}$.

Substituting (3) and (4) in (1) and (2) we get transformed equations of the form $F(P, Q) = 0$ and $F_1(x, P) = F_2(y, Q)$ and this can be easily solvable by standard method.

Type 6

Case 2. For $m = -1$ put

$$Z = \log z$$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} \Rightarrow P = \frac{1}{z}p \Rightarrow P = z^{-1}p \tag{5}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y} \Rightarrow Q = \frac{1}{z}q \Rightarrow Q = z^{-1}q, \tag{6}$$

where $P = \frac{\partial Z}{\partial x}$ and $Q = \frac{\partial Z}{\partial y.}$

Substituting (5) and (6) in (1) and (2) we have transformed equations of the form $F(P, Q) = 0$ and $F_1(x, P) = F_2(y, Q)$ and this can be easily solvable by standard method.

Example 40.

Solve $z^2(p^2+q^2)=x^2+y^2$.

Solution. *The given equation is of the form* $F_1(x, z^m p) = F_2(y, z^m q)$. *Putting* $Z = z^2$ *we get* $P = 2zp \quad \Rightarrow \quad \frac{P}{2} = zp$ *and* $Q = 2zq \Rightarrow \frac{Q}{2} = zq$. *Hence the equation becomes*

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x^2+y^2 \Rightarrow P^2+Q^2 = 4x^2+4y^2 \Rightarrow P^2-4x^2 = -Q^2+4y^2 = 4a^2 (say)$$

$$P^2 - 4x^2 = 4a^2 \quad and \quad -Q^2+4y^2 = 4a^2$$

$$dZ = Pdx + Qdy$$

$$\int dZ = 2\int \sqrt{x^2+a^2}dx + 2\int \sqrt{y^2-a^2}dy$$

$$Z = 2\left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)\right] + 2\left[\frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)\right]$$

$$z^2 = x\sqrt{x^2+a^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2-a^2} - a^2\cosh^{-1}\left(\frac{y}{a}\right).$$

Example 41.

Solve $(z^2p + x)^2 + (z^2q + y)^2 = 1.$
Solution. *The given equation is of the form* $F_1(x, z^m p) = F_2(y, z^m q)$. *Putting* $Z = z^3$, *we get* $P = 3z^2p \Rightarrow \frac{P}{3} = z^2p$ *and* $Q = 3z^2q \Rightarrow \frac{Q}{3} = z^2q$. *Hence the equation becomes*

$$\left(\frac{P}{3} + x\right)^2 + \left(\frac{Q}{3} + y\right)^2 = 1 \Rightarrow P = 3(a - x)$$

$$1 - \left(\frac{Q}{3} + y\right)^2 = a^2 \Rightarrow Q = 3(\sqrt{1 - a^2} - y)$$

$$dZ = Pdx + Qdy$$

$$\int dZ = \int 3(a - x)dx + \int 3(\sqrt{1 - a^2} - y)dy$$

$$z = 3\left(ax - \frac{x^2}{2}\right) + 3\left((\sqrt{1 - a^2})y - \frac{y^2}{2}\right) + c$$

$$z^3 = 3\left(ax - \frac{x^2}{2}\right) + 3\left((\sqrt{1 - a^2})y - \frac{y^2}{2}\right) + c.$$

References

- T. Amaranath, *An Elementary Course in Partial Differential Equations*, Second Edition, Narosa Publishing House, 1997.
- Ian Sneddon, *Elements of Partial Differential Equations*, McGraw-Hill Book Company, 1957.